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Solving Delay Differential Equations ByIman Transform Method

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ABSTRACT:
In this paper, we implement the Iman transform method for the solution of delay differential equations (DDEs). The method executes the DDEs by implementing its properties on the given DDE. Also, the method treats the nonlinear terms with a well posed formula. The method is easy to implement with high level of accuracy. Also, restricted transformations, perturbation, linearization or discretization are not recognized. The resulting numerical evidences show that the method Converges favorably to the analytic solution. All computational frameworks are performed with maple 18 software.

KEYWORDS: Iman transforms; Delay differential equation; Approximate solution; Partial derivatives.

Introduction

Delay differential equations appear in numerous applications in science and engineering, for example in bioscience, physics, chemistry, population dynamics, etc, and many can be found in [1]. In this paper, the Iman transform method is used to solve the delay differential equations (DDEs) of the form

$$y^{(n)}(\mathrm{t}) = f\big(t, y(t), y(t-\tau_i) \dots\dots y(t-\tau_i)\big), \quad t \geq 0 \qquad (1)$$
$$y(t) = g(t), \quad -\tau \leq t \leq 0$$

Where $g(t)$is the initial function, τ_i , $i > 0$, is called the delay or lag function ,f is a given function with $\tau_i \leq t$. If $\tau_i > 0$ is a constant , it is a constant dependent delay; if $\tau_i(x) \geq 0$ is time dependent , it is time dependent delay and if $u\big(\tau_i(x)\big) \geq 0$ is state dependent, it is state dependent delay.

In recent time, there have been keen interests in the solution of DDEs as it appears in the literature. Various numerical methods have been formulated and implemented for these particular equations. Popular numerical methods for DDEs include, the Adomian decomposition method [2-3], the variation iteration method [4], the differential transform method [5], the Runge – Kutta method [6], the Hermite interpolation method [7], the variable multistep method[8], the decomposition method [9], the direct block one step method[10], B- spline collocation method [11], the direct two and there point one-step block method [12], etc.

The Iman transform method [13-16], which is adopted in this paper is highly effective and reliable. The method executes the DDEs by simply implementing its properties on the given DDE. However, the nonlinear terms are treated with a well posed formula found in [13]. The initial approximation and recursion formula to compute the components $u_{n+1}(x), n \geq 0,$ are derived directly in the scheme execution. The approximate solution is written as the partial sum of the components $u_{n+1}(x), n \geq 0$For N, where N is an integer. The method is easy to implement with high level of accuracy. Restricted transformations, perturbation, linearization or discretization are not welcomed in this method.

This paper is organized as follows. Basic definitions and notations are presented in section two. The Iman transform method for the components nth order DDEs are given in section three. Numerical

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applications of the method to linear and nonlinear DDEs are considered in section four. Finally, the conclusion is given in section five.

Basic Definitions and Notations

i. Let $f(x)$ for$x \geq 0$, then the Iman transform ([13 – 16]) of $f(x)$ is a function of s defined by:

$$I[f(t)] = \frac{1}{s^2}\int_0^\infty f(x)e^{-s^2x}\,dx \qquad ,x \geq 0$$

ii. The Iman transform [13] of DDE derivative is obtained by integration by part. That is,

$$I[u'(x)] = r^2L(r) - \frac{u(0)}{r^2}$$

$$I[u''] = r^4L(r) - \frac{u'(0)}{r^2} - u(0)$$

Where

$$L_n(r) = r^{2n}K(r) - \sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{r^{4-2n+2k}}$$

iii. Some of the Iman transform properties can be found in the references [13-16], and are given as;

$$I[1] = \frac{1}{r^4}$$

$$I[x^n] = \frac{n!}{r^{2n+4}}$$

$$I^{-1}\left[\frac{n!}{r^{2n+4}}\right] = x^n$$

iv. Standard Iman transform for some special functions found in [13] are given blow in Table 1:

Table 1: Special function versus Iman transform equivalent

S.no	$f(t)$	$I\{f(t)\}$
1	1	$\frac{1}{r^4}$
2	t	$\frac{1}{r^6}$
3	e^{at}	$\frac{1}{r^2(r^2-a)}$
4	$\sin(at)$	$\frac{a}{r^2(r^4+a^2)}$
5	$\cos(at)$	$\frac{1}{r^4+a^2}$

Iman Transform Method

Let us consider the general nonlinear ordinary differential equation (ODE) of the form

$$\frac{d^mf(x)}{dx^m} + Rf(x) + Nf(x) = G(x), m = 1,2,3,4,\ldots \qquad (2)$$

With initial condition

$$\left.\frac{d^{m-1}f(x)}{dx^{m-1}}\right|_{x=0} = g_{m-1}(x), m = 1,2,3,4,\ldots$$

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$\frac{d^m f(x)}{dx^m}$ is the derivative of $f(x)$ of order m which is invertible, $Nf(x)$ is the nonlinear term, R is a linear operator and $G(x)$ is the source term.
Applying the Iman transform ([13-16]), we obtain

$$I\left[\frac{d^m f(x)}{dx^m}\right] + I[Rf(x) + Nf(x)] = I[G(x)] \quad (3)$$

By definition (ii), we have that

$$I\left[\frac{d^m f(x)}{dx^m}\right] = I[G(x)] - I[Rf(x) + Nf(x)]$$

$$r^{2m}L(r) - \sum_{n=0}^{m-1} \frac{1}{r^{4-2m+2n}} \frac{d^n f(0)}{dx^n} = I[G(x)] - I[Rf(x) + Nf(x)]$$

$$I[f(x)] = \sum_{n=0}^{m-1} \frac{1}{r^{4+2n}} \frac{d^n f(0)}{dx^n} + \frac{1}{r^{2m}} I[G(x)] - \frac{1}{r^{2m}} I[Rf(x) + Nf(x)] \quad (4)$$

Applying the Iman inverse operator, I^{-1} on both sides of (4) we obtain

$$f(x) = I^{-1}\left[\sum_{n=0}^{m-1} \frac{1}{r^{4+2n}} \frac{d^n f(0)}{dx^n}\right] + I^{-1}\left[\frac{1}{r^{2m}} I[G(x)]\right] - I^{-1}\left[\frac{1}{r^{2m}} I[Rf(x) + Nf(x)]\right] (5)$$

Where

$$\left.\frac{d^{m-1} f(x)}{dx^{m-1}}\right|_{x=0} = g_{m-1}(x), m = 1,2,3,4, \ldots$$

is the partial derivative of the initial condition.
By the Iman transformed method, equation (5) can be written as

$$\sum_{n=0}^{\infty} f_n(x) = I^{-1}\left[\sum_{n=0}^{m-1} \frac{1}{r^{4+2n}} \frac{d^n f(0)}{dx^n}\right] + I^{-1}\left[\frac{1}{r^{2m}} I[G(x)]\right] - I^{-1}\left[\frac{1}{r^{2m}} I\left[R \sum_{n=0}^{\infty} f_n(x) + \sum_{n=0}^{\infty} A_n(x)\right]\right] (6)$$

Where

$$A_n(x) = \sum_{r=0}^{n} f_r(x) f_{n-r}(x). \quad (7)$$

Comparing both sides of equation (6), we obtain

$$f_0(x) = I^{-1}\left[\sum_{n=0}^{m-1} \frac{1}{r^{4+2n}} \frac{d^n f(0)}{dx^n}\right] + I^{-1}\left[\frac{1}{r^{2m}} I[G(x)]\right]$$

$$f_1(x) = -I^{-1}\left[\frac{1}{r^{2m}} I[Rf_0(x) + A_0(x)]\right]$$

$$f_2(x) = -I^{-1}\left[\frac{1}{r^{2m}} I[Rf_1(x) + A_1(x)]\right]$$

$$\vdots$$

$$f_{n+1}(x) = -I^{-1}\left[\frac{1}{r^{2m}} I[Rf_n(x) + A_n(x)]\right], m = 1,2,3,4, \ldots, n \geq 0 \ (8)$$

Thus, the approximate solution can be written as

$$f(x) = \sum_{n=0}^{N} f_n(x), as\ N \to \infty. \quad (9)$$

Remark 3.1

In the case of nonlinear problems, equation (6) becomes

$$\sum_{n=0}^{\infty} f_n(x) = I^{-1}\left[\sum_{n=0}^{m-1} \frac{1}{r^{4+2n}} \frac{d^n f(0)}{dx^n}\right] + I^{-1}\left[\frac{1}{r^2} I[G(x)]\right] - I^{-1}\left[\frac{1}{r^2} I\left[R \sum_{n=0}^{\infty} f_n(x) + \sum_{n=0}^{\infty} A_n(x)\right]\right]$$

Such that by comparison, the components $f_n(x)$ becomes

$$f_0(x) = I^{-1}\left[\sum_{n=0}^{m-1} \frac{1}{r^{4+2n}} \frac{d^n f(0)}{dx^n}\right] + I^{-1}\left[\frac{1}{r^2} I[G(x)]\right]$$

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$$f_1(x) = -I^{-1}\left[\frac{1}{r^2} I[Rf_0(x) + A_0(x)]\right]$$
$$f_2(x) = -I^{-1}\left[\frac{1}{r^2} I[Rf_1(x) + A_1(x)]\right]$$
$$\vdots$$
$$f_{n+1}(x) = -I^{-1}\left[\frac{1}{r^2} I[Rf_n(x) + A_n(x)]\right]$$

Numerical Applications

In this section, the Iman transform method is applied to solve linear and nonlinear DDEs. Numerical results are compared with the variation iteration with He is polynomials [4] for same problems.

Example 4.1 [4]

Consider the following nonlinear delay differential equation (NDDE) of first order:

$\frac{df}{dx} = 1 - 2f^2\left(\frac{x}{2}\right), 0 \leq x \leq 1,$ (10)

With the initial condition

$$f(0) = 0.$$

The exact solution of the problem is

$$f(x) = \sin x$$

Solution

Applying the Iman transform on the both sides, we have

$$I\left[\frac{df(x)}{dx}\right] = I[1] - 2I\left[f^2\left(\frac{x}{2}\right)\right]$$

By definition (ii), we have

$$r^2 L(r) - \frac{f(0)}{r^2} = \frac{1}{r^4} - 2I\left[f^2\left(\frac{x}{2}\right)\right]$$

$I[f(x)] = \frac{1}{r^6} - \frac{2}{r^2} I\left[f^2\left(\frac{x}{2}\right)\right]$ (11)

Applying the Iman inverse operator, I^{-1} on both sides of (11) we obtain

$f(x) = I^{-1}\left[\frac{1}{r^6}\right] - 2I^{-1}\left[\frac{1}{r^2} I\left[f^2\left(\frac{x}{2}\right)\right]\right]$ (12)

By definition (iiic), we have $I^{-1}\left[\frac{1}{r^6}\right] = x$.

Hence

$f(x) = x - 2I^{-1}\left[\frac{1}{r^2} I\left[f^2\left(\frac{x}{2}\right)\right]\right]$ (13)

By the Iman transform method, equation (13) can be written as

$$f_0(x) = x,$$

$f_{n+1}(x) = -2I^{-1}\left[\frac{1}{r^2} I\left[f^2\left(\frac{x}{2}\right)\right]\right], n \geq 0,$ (14)

Where

$$A_n\left(\frac{x}{2}\right) = \sum_{r=0}^{n} f_r\left(\frac{x}{2}\right) f_{n-r}\left(\frac{x}{2}\right).$$

For $n = 0$, we have:

$$A_0\left(\frac{x}{2}\right) = \frac{x^2}{4}.$$

Which implies that

$$f_1(x) = -2I^{-1}\left[\frac{1}{r^2} I\left[\frac{x^2}{4}\right]\right] = -I^{-1}\left[\frac{1}{r^{10}}\right] = -\frac{x^3}{3!}.$$

For $n = 1$, we have:

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$A_1\left(\frac{x}{2}\right) = -\frac{x^4}{48}$.
Which implies that

$$f_2(x) = \frac{1}{24} I^{-1}\left[\frac{1}{r^2} I[x^4]\right] = I^{-1}\left[\frac{1}{r^{14}}\right] = \frac{x^5}{5!}.$$

For $n = 2$, we have:

$$A_2\left(\frac{x}{2}\right) = \frac{x^6}{1440}.$$

Such that

$$f_3(x) = -\frac{1}{720} I^{-1}\left[\frac{1}{r^2} I[x^6]\right] = -A^{-1}\left[\frac{1}{r^{18}}\right] = -\frac{x^7}{7!}.$$

For $n = 3$, we have:

$$A_3\left(\frac{x}{2}\right) = -\frac{x^8}{80640}.$$

Which gives

$$f_4(x) = \frac{1}{40320} I^{-1}\left[\frac{1}{r^2} I[x^8]\right] = I^{-1}\left[\frac{1}{r^{22}}\right] = \frac{x^9}{9!}.$$

Therefore the approximate solution is given as

$$f(x) = \sum_{n=0}^{N} f_n(x) = f_0(x) + f_1(x) + f_2(x) + f_3(x) + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sin x$$

It is obvious that for $n \geq 0$, the approximate solution converges rapidly to the analytic solution. This same result was equally obtained in [4] using variation iteration method with He's polynomials.

Example 4.2
Consider the following linear delay differential equation (NDDE) of the first order:
$\frac{df(x)}{dx} = \frac{1}{2} e^{\frac{x}{2}} f\left(\frac{x}{2}\right) + \frac{1}{2} f(x), \quad 0 \leq x \leq 1, \quad (15)$
With the initial condition

$$f(0) = 1.$$

The exact solution of the problem is

$$f(x) = e^x.$$

Solution
Applying the Iman transform on the both sides, we have:

$$I\left[\frac{df(x)}{dx}\right] = \frac{1}{2} \mathrm{I}\left[e^{0.5x} f\left(\frac{x}{2}\right) + f(x)\right]$$

By definition (ii) we have:
$r^2 L(r) - \frac{f(0)}{r^2} = \frac{1}{2} \mathrm{I}\left[e^{0.5x} f\left(\frac{x}{2}\right) + f(x)\right] \quad (16)$
$\mathrm{I}[\mathrm{f}(\mathrm{x})] = \frac{1}{\mathrm{r}^4} + \frac{1}{2} \mathrm{I} \frac{1}{\mathrm{r}^2}\left[\mathrm{e}^{0.5\mathrm{x}} \mathrm{f}\left(\frac{\mathrm{x}}{2}\right) + \mathrm{f}(\mathrm{x})\right]$
$f(x) = 1 + \frac{1}{2} I^{-1}\left[\frac{1}{r^2} I\left[e^{0.5x} f\left(\frac{x}{2}\right) + f(x)\right]\right] \quad (17)$
By the Iman transform method the equation above can be written as:

$$f_0(x) = 1.$$

$f_{n+1}(x) = \frac{1}{2} I^{-1}\left[\frac{1}{r^2} I\left[e^{0.5x} f_n\left(\frac{x}{2}\right) + f_n(x)\right]\right], \quad n \geq 0, \quad (18)$
For $n = 0$,

$$f_1(x) = \frac{1}{2} I^{-1}\left[\frac{1}{r^2} I[e^{0.5x} + 1]\right],$$

$= \frac{1}{2} I^{-1}\left[\frac{1}{r^2} I[e^{0.5x}] + \frac{1}{r^2} I[1]\right] \quad (19)$

But:

$$I[e^{0.5x}] = \frac{1}{r^4 - 0.5r^2} = \frac{1}{r^4}\left[\frac{1}{1 - \frac{0.5}{r^2}}\right]$$
$$= \frac{1}{r^4}\left[1 - \frac{0.5}{r^2}\right]^{-1}$$

Hence, equation (19) can be written as:

$$f_1(x) = \frac{1}{2}I^{-1}\left[\left[\frac{1}{r^6} + \frac{0.5}{r^8} + \frac{0.25}{r^{10}} + \frac{0.125}{r^{12}} + \cdots\right] + \frac{1}{r^6}\right]$$

By definition (iii c) we have:

$$f_1(x) = x + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \cdots$$

Following the above execution with the help of maple 18 software, we obtain the following approximation for $n \geq 1$

$$f_2(x) = \frac{3}{8}x^2 + \frac{13}{192}x^3 + \frac{13}{1024}x^4 + \cdots$$
$$f_3(x) = \frac{5}{64}x^3 + \frac{63}{4096}x^4 + \cdots$$

Thus the approximate becomes

$$\mathrm{f(x)} = \sum_{n=0}^{\infty} \mathrm{f_n(x)} = \mathrm{f_0(x)} + \mathrm{f_1(x)} + \mathrm{f_2(x)} + \mathrm{f_3(x)} + \cdots$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$$

Conclusion

The Iman transform method has been successively implemented for finding the solution of delay differential equations (DDES). The method converges rapidly to the analytic solution as shown in the examples. Hence, the method is accurate and more reliable in seeking the solution of DDEs.

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